12 Probability

INTRODUCTION

'It may rain today'.

'Rajesh is quite sure to top his class'.

'It is highly unlikely that Salman will marry Deepika'.

'I have bet 100 rupees on India winning against Pakistan'.

All the above statements indicate *uncertainties* in everyday life.

Probability is a measure of uncertainty.

The theory of probability developed as a result of studies of games of chance or gambling. Suppose you pay ₹2 to draw a card out of a pack of 52 cards—if you draw an ace, you get ₹20, otherwise you get nothing. Should you play such a game? Does it give a *fair* chance of winning, or are you going to lose in the long run? Search for mathematical answers to these types of questions led to the development of the modern theory of probability. Italian mathematician Jerome Cardan, French mathematicians Pascal and Pierre de Fermat, Swiss mathematician James Bernouli, and Russian mathematician Kolmogorov were the pioneers in these studies.

RANDOM EXPERIMENTS AND SAMPLE SPACES

Experiment

An action (or operation) which results in some (well-defined) outcomes is called an ***experiment.***

Sometimes, the result (*i.e.* outcome) is unique. For example, if any triangle is given then without knowing the three angles, we can definitely say that the sum of measures of the three angles is 180°. Such experiments are called **deterministic experiments.**

Sometimes the outcome may not be unique. For example, tossing a coin may throw up either heads or tails. Such experiments are called **probabilistic** or **random.**

Random experiment

An experiment is called ***random*** *if it results in two or more outcomes and it is not possible to tell (predict) the outcome in advance.*

Thus, an experiment is called ***random*** *if it satisfies the following two conditions :*

(*i*) *It has more than one possible outcome.*

(*ii*) *It is not possible to tell (predict) the outcome in advance.*

For example :

(*i*) tossing a coin

(*ii*) tossing two coins simultaneously

(*iii*) tossing a coin three times

(*iv*) throwing a die

(*v*) drawing a card from a pack of 52 (playing) cards.

All these are random experiments.

From now onwards, whenever the word experiment is used it will mean random experiment.

Sample Space

The collection of all possible outcomes of a random experiment is called **sample space** *associated with it, it is usually denoted by S. Each element of the sample space (set) is called a* **sample point.**

For example :

(*i*) When we toss a coin once, it may come up in either of two ways—heads or tails. So there are two possible outcomes of this random experiment. If we denote heads by H and tails by T, then the sample space of this experiment is

$$S = \{H, T\}$$

(*ii*) When we throw (or roll) a die, it may land with any of its 6 faces pointing upward. Thus, the outcome of this experiment is getting any of the six numbers 1, 2, 3, 4, 5, 6. Hence, the sample space for this experiment is

$$S = \{1, 2, 3, 4, 5, 6\}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Describe the sample space of a random experiment when a coin and a die are tossed together.*

Solution. Sample space S consists of 2×6 *i.e.* 12 outcomes.

$$S = \{(H, 1), (H, 2), (H, 3), (H, 4), (H, 5), (H, 6), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}.$$

It can also be written as

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}.$$

Example 2. *Find the sample space associated with the experiment of rolling a pair of dice (one is blue and the other red) once. Also find the number of elements of the sample space.*

Solution. The blue die may come up with any of the numbers 1, 2, 3, 4, 5, 6, and the red die may also come up with any of the numbers 1, 2, 3, 4, 5, 6. Together, we may denote the outcome by an ordered pair (x, y) where x appears on the blue die and y appears on red die. Then, the sample space is given by

$$S = \{(x, y)\,;\, x \text{ is a number on the blue die, and } y \text{ is a number on the red die}\}$$

We can list all possible outcomes as:

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Number of elements (outcomes) of the sample space is 6×6 *i.e.* 36.

Example 3. *We wish to choose one child out of 2 boys and 3 girls. A coin is tossed. If it comes up heads, a boy is chosen, otherwise a girl is chosen. Describe the sample space.*

Solution. The outcome of toss of a coin may be denoted by H or T. The boys may be called B_1, B_2 and the girls G_1, G_2, G_3. Then the sample space is

$$S = \{HB_1, HB_2, TG_1, TG_2, TG_3\}.$$

Example 4. *An experiment consists of recording boy-girl composition of families with 2 children.*

(*i*) *What is the sample space if we are interested in knowing whether it is a boy or a girl in the order of their births?*

(*ii*) *What is the sample space if we are interested in number of girls in the family?*

Solution. (*i*) As there are 2 children, there are 4 possible outcomes—

boy-boy, boy-girl, girl-boy, girl-girl. Hence, the sample space is

S = {BB, BG, GB, GG}.

(*ii*) As we are interested in knowing only the number of girls, there are 3 possible outcomes—no girl, 1 girl, 2 girls. Hence, the sample space is

S = {0, 1, 2}.

Example 5. *A bag contains 2 red and 3 black balls. What is the sample space when the experiment consists of drawing?*

(*i*) *1 ball* (*ii*) *2 balls (assuming that order is not important)*

Solution. Let the red balls be denoted by r_1, r_2 and the black balls by b_1, b_2, b_3.

(*i*) When one ball is drawn, the sample space is $\{r_1, r_2, b_1, b_2, b_3\}$.

(*ii*) When two balls are drawn (order not important), then the sample space is

$\{r_1 r_2, r_1 b_1, r_1 b_2, r_1 b_3, r_2 b_1, r_2 b_2, r_3 b_3, b_1 b_2, b_1 b_3, b_2 b_3\}$.

Example 6. *A bag contains one red ball and four identical black balls. What is the sample space when the experiment consists of*

(*i*) *drawing one ball?*

(*ii*) *drawing two balls one by one (assuming that after drawing one ball, it is replaced before drawing the second ball)?*

Solution. Let the red ball be denoted by R and each identical black ball by B.

(*i*) When one ball is drawn, the sample space is {R, B}.

(*ii*) When two balls are drawn one by one (assuming that after drawing one ball, it is replaced before drawing the second ball), then the sample space is {RR, RB, BR, BB}.

Example 7. *A girl has a one rupee coin, a two rupee coin and a five rupee coin in her pocket. She takes out two coins of her pocket, one after the other. Describe the sample space.*

Solution. Let one rupee, two rupee and five rupee coins be denoted by O, T and F respectively.

The sample space of the given experiment is

S = {OT, OF, TF, FT, FO, TO}.

Example 8. *One die of red colour, one of white colour and one of blue colour are placed in a bag. One die is selected at random and rolled, its colour and the number on its uppermost face is noted. Describe the sample space.*

Solution. Let die of red colour, white colour and blue colour be denoted by R, W and B respectively.

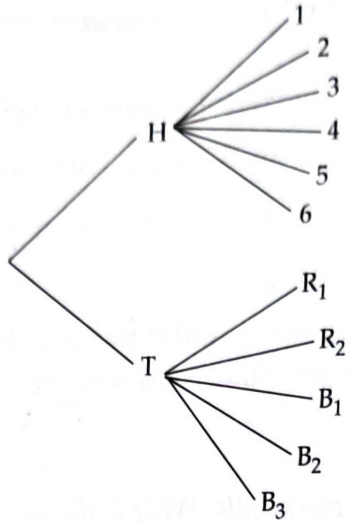
The sample space of the given experiment is

{R1, R2, R3, R4, R5, R6, W1, W2, W3, W4, W5, W6, B1, B2, B3, B4, B5, B6}.

Example 9. *A coin is tossed. If it shows a tail, we draw a ball from a box which contains 2 red and 3 black balls. If it shows head, we throw a die. Find the sample space for this experiment.*

Solution. Let us denote the red balls by R_1, R_2 and the black balls by B_1, B_2, B_3.

We can draw the following tree diagram for the given experiment :



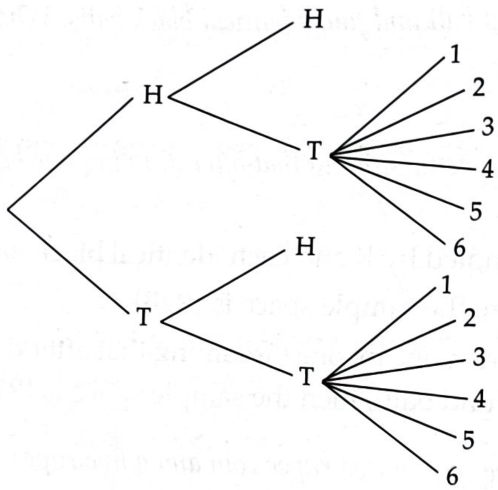
Hence, the sample space of the given experiment is

{H1, H2, H3, H4, H5, H6, TR_1, TR_2, TB_1, TB_2, TB_3}.

There are 11 possible outcomes.

Example 10. *A coin is tossed twice. If the second throw results in a tail, we roll a die. Describe the sample space. How many outcomes are possible in this experiment?*

Solution. We can draw the following diagram for the given experiment :



Hence, the sample space is

S = {HH, HT1, HT2, HT3, HT4, HT5, HT6, TH, TT1, TT2, TT3, TT4, TT5, TT6}.

There are 14 possible outcomes.

EXERCISE 12.1

1. Describe the sample space for the following experiments:
   (i) Two coins are tossed.
   (ii) One coin is tossed twice.
   (iii) Three coins are tossed simultaneously.
   (iv) A coin is tossed, and if head comes up, a die is thrown.
   (v) A card is drawn from a deck of playing cards, and its colour is noted.
   (vi) A person is noting down the number of accidents along a busy road during a year.
2. A box contains 1 red and 3 identical white balls. Two balls are drawn one by one without replacement. Write the sample space for this experiment.
3. Consider the experiment in which a coin is tossed repeatedly until a tail comes up for the first time. Describe the sample space.

4. In team A, there are 2 boys and 2 girls. In team B, there is one boy and 3 girls. First a team is chosen, and then a participant. Describe the sample space.
5. A coin is tossed. If it shows head, we draw a ball from a bag containing 3 blue and 4 white balls; if it shows a tail, we throw a die. Describe the sample space of the experiment.
6. Suppose 3 bulbs are selected at random from a lot. Each bulb is tested and classified as defective (D) or non-defective (N). Write the sample space of this experiment.
7. The numbers 1, 2, 3 and 4 are written separately on four slips of paper. The slips are then put in a box and mixed thoroughly. A person draws two slips from the box, one after the other, without replacement. Describe the sample space of the experiment.

Answers

1. (*i*) {HH, HT, TH, TT} (*ii*) {HH, HT, TH, TT}
   (*iii*) {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}
   (*iv*) {H1, H2, H3, H4, H5, H6, T} (*v*) {Spade, Heart, Diamond, Club}
   (*vi*) {0, 1, 2, 3, ... }
2. {RW, WR, WW} 3. {T, HT, HHT, HHHT, ...}
4. {Ab_1, Ab_2, Ag_1, Ag_2, Bb_3, Bg_3, Bg_4, Bg_5},
   where team A has two boys b_1, b_2 and two girls g_1, g_2, while team B has one boy b_3, and three girls g_3, g_4, g_5
5. {HB_1, HB_2, HB_3, HW_1, HW_2, HW_3, HW_4, T1, T2, T3, T4, T5, T6}
   where B_1, B_2, B_3 denote blue balls and W_1, W_2, W_3, W_4 denote white balls
6. {DDD, DDN, DND, DNN, NDD, NDN, NND, NNN}
7. {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (2, 4), (3, 1), (3, 2), (3, 4), (4, 1), (4, 2), (4,3)}

EVENTS

Event

A subset of the sample space associated with a random experiment is called an ***event.***

For example :

(*i*) When a die is thrown, we can get any number 1, 2, 3, 4, 5, 6. So the sample space is S = {1, 2, 3, 4, 5, 6}. There are a variety of ways in which the outcome could be reported:

Getting a six, corresponding to {6}.

Getting an even number, corresponding to {2, 4, 6}.

Getting an odd number, corresponding to {1, 3, 5}.

Getting a prime number, corresponding to {2, 3, 5}.

Getting a number less than 5, corresponding to {1, 2, 3, 4}, and so on.

All these are the events of the random experiment of throwing a die.

(*ii*) When a pair of coins is tossed, the sample space is

S = {HH, HT, TH, TT}. A few events for this could be:

Getting exactly two heads : {HH}

Getting exactly one head : {HT, TH}

Getting no head : {TT}

Getting atleast one head : {HH, HT, TH}, etc.

Occurrence of an event

When the outcome of an experiment satisfies the condition mentioned in the event, then we say that ***event has occurred.***

For example :

(*i*) In the experiment of throwing a die, an event E may be defined as 'getting an even number'. If the die comes up with any of the numbers 2, 4 or 6, we say that event E has occurred, otherwise, if we get 1, 3 or 5, we say that event E has not occurred.

(*ii*) In the experiment of tossing a pair of coins, an event E may be getting two heads. If the pair of coins come up with two heads, then we say that event E has occurred, otherwise, we say that event E has not occurred.

Types of events

Simple event *has only one sample point (outcome) of the sample space.* For example, if 3 coins are tossed up, {HHH}, {TTT} are simple events. *A simple event is also called* ***elementary event*** *or* ***indecomposible event.***

Compound event *has two or more than two sample points (outcomes) of the sample space.*

For example, when 3 coins are tossed, {HHT, HTH, THH, HHH} is a compound event corresponding to the statement "getting atleast two heads". *A compound event is also called* ***decomposible event.***

Sure event *is the sample space S itself.* For example, when we throw a die, S = {1, 2, 3, 4, 5, 6}. Then {getting a number less than 7} is a sure event whereas {getting a number less than 5} is not a sure event.

Impossible event *corresponds to null set* ϕ. For example, in the case of dice, {getting a number greater than 6} is an impossible event.

Equally likely outcomes. *If there is no reason for any one outcome to occur in preference to any other outcome, then we say that outcomes are equally likely.* For example, in drawing a card from well shuffled pack, all 52 possible outcomes are equally likely. Getting a spade is as likely as getting a heart (there are 13 cards of each); getting a king is as likely as getting a queen (there are 4 cards of each).

Algebra of events

Since events can be represented as sets, the set operations like union, intersection, complement, difference etc. can be applied to events.

Complement of an event

The ***complement of an event*** *E, denoted by* $\overline{E}$ *or E′ or* E^c, *is the set of all outcomes of the sample space other than the outcomes occurring in E.*

In other words, $E^c = \{w : w \in S \text{ and } w \notin E\} = S - E$.

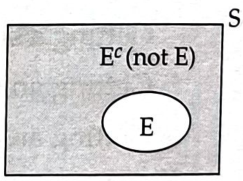
Note that $E \cup E^c = S$ and $E \cap E^c = \phi$.

Take the example of tossing two coins. The sample space is

S = {HH, HT, TH, TT}.

If E is the event "atleast one tail appears" = {HT, TH, TT}, then

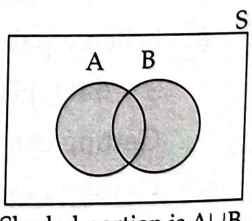
E^c or "not E" or $\overline{E}$ or E′ = {HH} = {no tail appears}.



The event "A or B"

Recall that union of two sets A and B, denoted by $A \cup B$, also called "A or B" is the set which contains all those elements which are either in A or in B or in both.

Similarly, event "A or B" is $A \cup B = \{w : w \in A \text{ or } w \in B\}$.



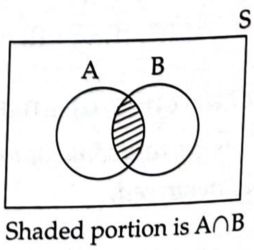
Shaded portion is $A \cup B$

The event "A and B"

Intersection of two sets A and B, denoted by $A \cap B$, also called "A and B" is the set of all those elements which are common to A and B *i.e.* which belong to both A and B.

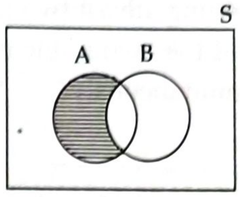
Similarly, event "A and B" is

$A \cap B = \{w : w \in A \text{ and } w \in B\}$.



Shaded portion is $A \cap B$

The event A but not B



Shaded portion is
$A - B = A \cap B'$

The difference of two sets A and B, denoted by $A - B$ is the set of those elements which are in A but not in B.

Similarly, event A but not B is

$A - B = \{w : w \in A \text{ and } w \notin B\} = A \cap B'$.

Exhaustive events

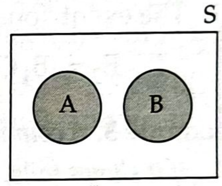
The events $E_1, E_2, \ldots, E_n$ associated with a random experiment with sample space S are called ***exhaustive*** *if $E_1 \cup E_2 \cup \ldots \cup E_n = S$.*

For example, if $S = \{1, 2, 3, 4, 5, 6\}$, $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$, $C = \{1, 2, 3, 4\}$, then $A \cup B = S$, so A and B are exhaustive events. But $A \cup C = \{1, 2, 3, 4, 5\} \neq S$, so A and C are not exhaustive events.

Note that when events $E_1, E_2, \ldots, E_n$ are exhaustive, then (atleast) one of them must necessarily occur every time the experiment is performed.

Mutually exclusive events

Two or more events associated with a random experiment are called ***mutually exclusive*** *if the occurrence of any one of them excludes the occurrence of the other events.*

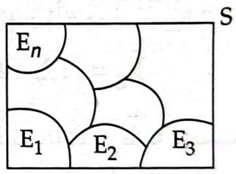


If A and B are mutually exclusive, then $A \cap B = \phi$ (the null set). In above example, $A \cap B = \{1, 3, 5\} \cap \{2, 4, 6\} = \phi$, so A and B are mutually exclusive, whereas $A \cap C = \{1, 3, 5\} \cap \{1, 2, 3, 4\} = \{1, 3\} \neq \phi$, so A and C are not mutually exclusive. Similarly in a deck of cards, let A = {getting a spade}, B = {getting a heart}, C = {getting a king}, then A and B are mutually exclusive as $A \cap B = \phi$, whereas A and C are not mutually exclusive as $A \cap C$ = {king of spade} $\neq \phi$.

Remark

The simple events of a sample space are always mutually exclusive.

Mutually exclusive and exhaustive events



Let S be the sample space associated with a random experiment, then the events $E_1, E_2, \ldots, E_n$ of S are called ***mutually exclusive and exhaustive*** *if $E_1 \cup E_2 \cup \ldots \cup E_n = S$ i.e. $\bigcup_{i=1}^{n} E_i = S$ and $E_i \cap E_j = \phi$ for all $i \neq j$.*

For example, let A = {getting a spade}, B = {getting a club}, C = {getting a heart}, D = {getting a diamond}, then A, B, C, D, are mutually exclusive and exhaustive events.

Exhaustive number of cases

It is the total number of possible outcomes of an experiment. Thus, when a coin is thrown, the exhaustive number of cases is 2 as we may get either heads or tails. When a die is thrown, the exhaustive number of cases is 6. When three dice are thrown simultaneously, the exhaustive number of cases is 6^3 *i.e.* 216. When 2 cards are drawn simultaneously from a pack of playing cards, the exhaustive number of cases is ${}^{52}C_2$, and so on.

Favourable number of cases

The number of cases favourable to an event is known as the favourable number of cases. Thus if the experiment is simultaneous throw of three coins, the number of cases favourable to the event 'getting atleast two heads' is 4 as the cases HHH, HHT, HTH, THH are favourable to it. Can you find the favourable number of cases for the event 'getting a sum of 10' when two dice are thrown simultaneously?

ILLUSTRATIVE EXAMPLES

Example 1. *A die is rolled. Let E be the event 'die shows 4' and F be the event 'die shows even number'. Are the events E and F mutually exclusive?*

Solution. When a die is rolled, sample space S = {1, 2, 3, 4, 5, 6}.

Event E = {4} and event F = {2, 4, 6}.

Here $E \cap F = \{4\} \neq \phi \Rightarrow$ events E and F are not mutually exclusive.

Example 2. *From a group of 3 boys and 2 girls, we select two children. Describe the sample space. How would you represent the events "both selected children are boys" and "one boy and one girl is selected"?*

Solution. Two children out of five can be selected in $^5C_2 = 10$ ways. If the three boys are represented by B_1, B_2, B_3 and the two girls are represented by G_1, G_2, then the sample space is

$$S = \{B_1B_2, B_1B_3, B_1G_1, B_1G_2, B_2B_3, B_2G_1, B_2G_2, B_3G_1, B_3G_2, G_1G_2\}.$$

The event "both selected children are boys" can be represented as

$$E_1 = \{B_1B_2, B_1B_3, B_2B_3\}$$

The event "one boy and one girl is selected" can be represented as

$$E_2 = \{B_1G_1, B_1G_2, B_2G_1, B_2G_2, B_3G_1, B_3G_2\}$$

Example 3. *A coin is tossed. If it shows head, we draw a ball from a bag consisting of 3 red and 4 black balls; if it shows tails, we throw a die. What is the sample space of this experiment? How would you represent the events "coin came up head" and "the throw of the die resulted in an even number"?*

Solution. If we denote the red balls by r_1, r_2, r_3 and the black balls by b_1, b_2, b_3, b_4, then the sample space is

$$S = \{Hr_1, Hr_2, Hr_3, Hb_1, Hb_2, Hb_3, Hb_4, T1, T2, T3, T4, T5, T6\}$$

The event "coin came up head" can be represented as

$$E_1 = \{Hr_1, Hr_2, Hr_3, Hb_1, Hb_2, Hb_3, Hb_4\}$$

The event "the throw of the die resulted in even number" can be represented as

$$E_2 = \{T2, T4, T6\}.$$

Example 4. *A coin is tossed three times. Consider the following events :*

A : 'No head appears', B : 'exactly one head appears', C : 'Atleast two heads appear'.

Do they form a set of mutually exclusive and exhaustive events?

Solution. The sample space of the experiment is

S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT},

A = {TTT}, B = {HTT, THT, TTH}, C = {HHT, HTH, THH, HHH}

Now $A \cup B \cup C$ = {TTT, HTT, THT, TTH, HHT, HTH, THH, HHH} = S,

So A, B and C are exhaustive events.

Also $A \cap B = \phi$, $A \cap C = \phi$ and $B \cap C = \phi$,

so events are pairwise disjoint *i.e.* they are mutually exclusive.

Hence A, B and C form a set of mutually exclusive and exhaustive events.

EXERCISE 12.2

1. Give an example of an impossible event.
2. Give an example of a sure event.
3. A coin is tossed. Write its sample space and all possible events.
4. Consider the experiment of rolling a die. If A be the event 'getting a prime number' and B be the event 'getting an odd number', then write the sets representing the events
   (*i*) A or B (*ii*) A and B (*iii*) A but not B (*iv*) not A.
5. A coin and a die are tossed. Describe the sample space and the following events :
   (*i*) A = getting a head and an even number (*ii*) B = getting a prime number.
   (*iii*) C = getting a tail and an odd number (*iv*) D = getting a head or a tail.
   Now answer the following questions :
   (*a*) Are A and B mutually exclusive? (*b*) Are A and C mutually exclusive?
   (*c*) Are A and C exhaustive? (*d*) Are A and D exhaustive?
6. Two dice are thrown. If E is the event "both dice come up with same number" and F is the event "product of the numbers on the two dice is odd", then describe
   (*i*) E (*ii*) F (*iii*) E or F (*iv*) E and F (*v*) E but not F.
7. From a group of 2 men and 3 women, two persons are selected. Describe the sample space of the experiment. If E is the event in which one man and one woman are selected, then which are the cases favourable to E?
8. Two dice are rolled. Let A, B and C be the events of getting a sum 2, sum 3, and a sum 4, respectively.
   (*i*) Is event A simple? (*ii*) Is event B simple?
   (*iii*) Is event C compound? (*iv*) Are events A and B mutually exclusive?
9. From a group of 2 boys and 3 girls, two children are selected at random. Describe the events
   (*i*) A : both selected children are girls.
   (*ii*) B : the selected group consists of one boy and one girl.
   (*iii*) C : atleast one boy is selected.
   Which pair (s) of events is (are) mutually exclusive?

Answers

1. E = {getting a number higher than 6} when a die is rolled
2. E = {getting a number less than 7} when a die is rolled
3. S = {H, T}. There are four possible events— {H, T}, {H}, {T}, ϕ
4. (*i*) {1, 2, 3, 5} (*ii*) {3, 5} (*iii*) {2} (*iv*) {1, 4, 6}
5. S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}
   (*i*) A = {H2, H4, H6} (*ii*) B = {H2, H3, H5, T2, T3, T5}
   (*iii*) C = {T1, T3, T5} (*iv*) D = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}
   (*a*) No, as $A \cap B = \{H2\} \neq \phi$ (*b*) Yes, as $A \cap C = \phi$
   (*c*) No, as $A \cup C \neq S$ (*d*) Yes, as $A \cup D = S$
6. (*i*) E = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)}
   (*ii*) F = {(1, 1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1), (5, 3), (5, 5)}
   (*iii*) E or F = $E \cup F$ = {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6), (1, 3), (1, 5), (3, 1), (3, 5), (5, 1), (5, 3)}